International Contest-Game MATH KANGAROO
Canada, 2007

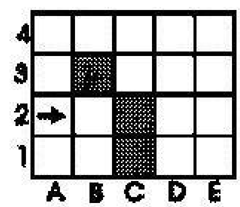
Grade 7 and 8
Solutions

Part A: Each correct answer is worth 3 points.

1. What is the value of the expression $\frac{2007}{2+0+0+7}$?

A) 1003 B) 75 C) 223 D) 213 E) 123

Solution: 2007/9=223
Answer: C



2. A robot starts walking on the table from square A2 at the direction of the arrow, as shown on the picture. It always goes forward. If it reaches a barrier, it always turns right. The robot will stop if he cannot go forward after turning right. On which square will it stop?

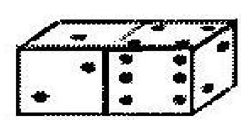
A) B2 B) A1 C) E1 D) D1 E) It will never stop

Solution: The robot follows the way: A2-B2-B1-A1-A2-A3-A4-B4-C4-**D4-E4**…E1-D1-..**D4** point from which repeat endlessly the sequence of squares.
Answer: E

3. Rose bushes were planted in a row, 2 m apart, on both sides of a road. How many bushes were planted along 20 m of the road?

A) 22 B) 20 C) 12 D) 11 E) 10

Solution: On one of the sides, along 20 m we have 11 roses (planted at 0, 2, 4, .., 20 meters), so on both sides we have 22.
Answer: A



4. A regular die has a total of 7 points on any two of its opposite faces. On the figure, two regular dice are placed, as shown. What is the sum of the points on all invisible faces of the dice?

A) 15 B) 12 C) 7 D) 27 E) another answer

Solution: On the right die, the numbers on the invisible faces are 1, 3, and 5. On the left die, the numbers on the invisible faces are 5, 6, 4, and 3. The total of all these numbers is 27.

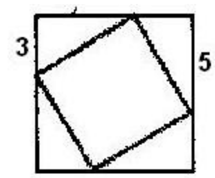
Alternatively, the total sum of all points on both dice is 7×3×2=42. The sum of the points we can see is 1+2+2+4+6=15, so the points on the invisible faces are 42-15=27.
Answer: D

5. The points A(2006, 2007), B(2007, 2006), C(-2006, -2007), D(2006, -2007), and E(2007, -2006) are plotted on a co-ordinate grid. Which of the segments is horizontal?

A) AD B) BE C) BC D) CD E) AB

Solution: Points with the same y-coordinate are on a horizontal line. So, the segment CD is horizontal.
Answer: D

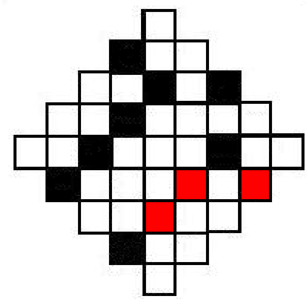
6. A small square is inscribed in a big one, as shown in the figure. The lengths of two of the segments are given (3 units and 5 units). What is the area (in square units) of the small square?



A) 16 B) 28 C) 34 D) 36 E) 49

Solution: The four triangles around the square are congruent, so, the side of the big square is 8 cm long, and its area is 64 square units. The area of each triangle is (3×5)/2=7.5 square units, hence, the area of the four triangles is 30 square units. Then, the area of the small square is 64-30=34 square units.
Answer: C

7. The figure on the right is composed of white and black unit squares. What is the least number of white squares to paint black for the figure to obtain a line of symmetry?



A) 4 B) 6 C) 5 D) 2 E) 3

Solution: The figure has 4 axes of symmetry. By trying out we can see that the smallest number of white squares to be coloured black is 3, as seen on the diagram.

Answer: E

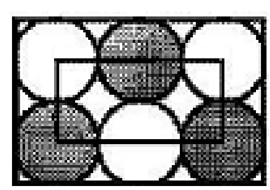
8. A number is called "palindrome" if it reads the same backwards as forwards. For example, 13931 is a palindrome. What is the difference between the least 5-digit palindrome number and the greatest 6-digit palindrome number?

A) 989989 B) 989998 C) 998998 D) 999898 E) 999988

Solution: The least 5-digit palindrome number is 10001, and the greatest 6-digit palindrome number is 999999, so the difference is 999999-10001=98998.
Answer: B

<u>Part B: Each correct answer is worth 4 points.</u>

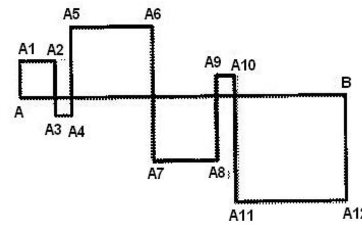
9. Six identical circles are arranged, as shown on the figure. The circles touch he sides of a large rectangle as well as each other. The vertices of the small rectangle coincide with the centres of four of the circles. The perimeter of the small rectangle is 60 cm. What is the perimeter of the large rectangle?



A) 160 cm B) 140 cm C) 120 cm D) 100 cm E) 80 cm

Solution: The perimeter of the small rectangle is composed of 12 circle radii, so, each radius is 60/12=5 cm. The perimeter of the big rectangle is composed of 20 radii, so its total length is 20×5=100 cm.
Answer: D

10. The squares on the figure are formed by intersecting the segment AB by the broken line $AA_1A_2...A_{12}B$. The length of AB is 24 cm. What is the length of the broken line $AA_1A_2...A_{12}B$?



A) 48 cm B) 72 cm C) 96 cm D) 56 cm E) 106 cm

Solution: The broken line consists of three sides from each of the squares, so its length will be ¾ of the total sum of their perimeters. On the other hand, the segment AB consists of one side of each square, so the length of AB is ¼ of the total sum of their perimeters. Hence, the length of the broken line is 3(AB)=72cm.
Answer: B

11. If **x** denotes any negative integer number, which of the following expressions will always have the greatest value?

A) $x+1$ B) $2x$ C) $-2x$ D) $6x+2$ E) $x-2$

Solution: Since x is a negative integer number, then the expressions 2x, 6x+2, and x-2 are always negative, while the expression x+1 is negative or zero. The only expression that is always positive is (-2x), hence, this is the expression with the greatest value, for any such number x.
Answer: C

12. Six points are chosen on two parallel lines *x* and *y,* as follows: 4 points are on line *x* and two points are on line *y*. How many triangles with their vertices among the given points are there?

A) 6 B) 8 C) 12 D) 16 E) 18

Solution: First, let us count triangles that have exactly one vertex on the line y. For each point on line y, we have 6 ways of choosing two points from the ones on line x. There are two points on line y, so the total number of triangles with exactly one of vertices on line y is 6×2=12.
Next, let us count triangles with exactly two vertices on the line y. We only need to choose one point on line x, for the third vertex. This can be done in 4 ways, so, there are 4 triangles in this category.
In conclusion, the total number of triangles with vertices among the given points is 12+4=16
Answer: D

13. Five integer numbers are written around a circle in a way that no two or three adjacent numbers have a sum divisible by 3. How many of these five numbers are divisible by 3?
14.
A) 0 B) 1 C) 2 D) 3 E) Impossible to determine

Solution: The condition that the sum of any two adjacent numbers is not divisible by 3 translates into the fact that the remainders of these numbers when divided by 3 cannot be a sequence of 1 and 2, and cannot be two zeroes. So, two adjacent numbers can only have remainders (0 ,1), (1, 0), (0, 2), (2, 0), (1, 1) or (2, 2). On the other hand, the sum of any three adjacent numbers is also not divisible by 3, therefore, the combinations (0, 0, 0), (1, 1, 1), (1, 0, 2), or (2, 0, 1) are not possible.

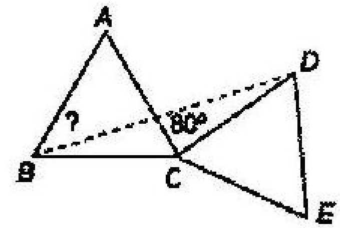
Let us assume that there is no multiple of 3 among the five numbers. It follows that all five remainders are either 1 or 2, and evidently, they cannot be arranged to comply with the above restrictions.

Let us assume that there is only one multiple of 3. Then, the only possibilities for the numbers adjacent to it, are either 1 and 1 or 2 and 2. Without loss of generality, let us assume these numbers are 1 and 1. So, we already have a sequence of 1, 0, 1 and we must add two more numbers to the right of it. These numbers cannot be (1,1), or (2, 1) or (1, 2) or (2, 2) or (0, 2) or (2, 0). The possibilities are (1, 0) or (0, 1) only. Therefore, we will need two multiples of 3.

If we have three or more multiples of 3 among these five numbers, there always will be three of them adjacent to each other, which is not allowed.

Answer: C

15. In the figure, ABC and CDE are congruent equilateral triangles. If the measure of the angle ACD is 80°, what is the measure of angle ABD?



A) 25° B) 30° C) 35° D) 40° E) 45°

Solution: The triangle BCD is isosceles (BC=CD), and the angle BCD = 60°+80°=140°, so the angle DBC is equal to (180°-140°)/2=20. In consequence, angle ABD is 60°-20°=40°.

Answer: D

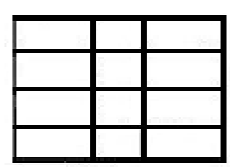
16. What percent of all natural numbers from 1 to 10000 are perfect squares?
(*Perfect square is a number that can be presented as a square of a natural number, for instance 100 = 10^2*).

A) 1% B) 1.5% C) 2% D) 2.5% E) 5%

Solution: $1=1^2$. $10000=100^2$. Since the squares of all numbers between 1 and 100 must be some numbers in the interval between 1^2and100^2, out of the natural numbers up to 10000 exactly 100 numbers are perfect squares, which is 100/10000=1/100=1%.

Answer: A

17. By drawing 9 lines (5 horizontal and 4 vertical) Peter can construct a table with 12 cells. If he had used 6 horizontal and 3 vertical lines, he would have constructed a table with 10 cells only. At most how many cells will there be in a table constructed by a total of 15 lines?



A) 22 B) 30 C) 36 D) 40 E) 42

Solution: It is easy to check that, for a given total number of lines, the maximum number of cells occurs when the numbers of the horizontal and vertical lines are as close to each other as possible, i.e., when the difference between them is the least possible. If the total number of lines is 15, the numbers 7 and 8 are such a pair. Then, the table will have 6 rows and 7 columns (or vice versa), or 42 cells in all.

Answer: E

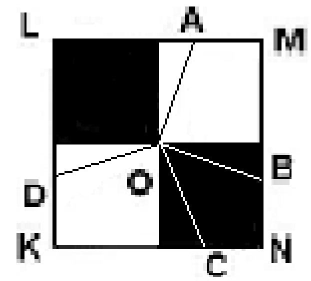
Part C: Each correct answer is worth 5 points.

18. A survey found that in 2006, 2/3 of all customers preferred product A, and 1/3 of all customers preferred product B. After a media campaign that promoted product B, a new survey in 2007 showed that ¼ of the customers who previously preferred product A are now buying product B. Which of the following statements is definitely true?

A) *5/12 of the customers buy product A, 7/12 buy product B.*
B) *1/4 of the customers buy product A, 3/4 buy product B.*
C) *7/12 of the customers buy product A, 5/12 buy product B.*
D) *1/2 of the customers buy product A, 1/2 buy product B.*
E) *1/3 of the customers buy product A, 2/3 buy product B.*

Solution: After the campaign, the fraction of all customers that represents the number of customers who prefer B is 1/3+(1/4 of (2/3))=1/2 . Therefore, *1/2 of the customers buy product A, 1/2 buy product B.*
Answer: D.

19. The segments OA, OB, OC, and OD are constructed from the centre O of the square KLMN to its sides, so that OA⊥OB and OC⊥OD (see the figure). The side of the square is 2. What is the total area of the shaded regions?



A) 1 B) 2 C) 2.5 D) 2.25
E) depends on the choice of the points B and C

Solution: Construct perpendicular segments from the centre of the square O to each of the sides. These segments will form four congruent right-angled triangles. (The triangles are congruent by ASA postulate, having one side – the perpendicular, and two angles: the right angle and the acute angles at O equal). Two of them will be black, while the other two (at the points B and C) will be white. Imagine that we rearrange the “black” area by removing the “black” triangles (with one vertex A and D) and replacing the white ones (with one vertex B and C) by them. This way, the shaded region will be exactly one half of the area of the original square. Hence, its area will be ½ (2×2)=2.
Answer: B

20. A broken calculator does not display the digit 1. For example, if we type in the number 3131, only the number 33 is displayed, with no spaces. Mike typed a 6-digit number into this calculator, but only 2007 appeared on the display. How many numbers could have Mike typed?

A) 12 B) 13 C) 14 D) 15 E) 16

Solution: According to the information given, two digits 1 are missing from the number entered by Mike. There are six possible positions for the digits. Every choice of two of them to be filled by 1s could be a possible entry, provided that in the remaining four positions Mike entered the digits 2, 0, 0, and 7, in this order. Therefore, the original question becomes equivalent to another question: in how many different ways can we choose two positions out of six positions? For the first position, we have six choices (it can be any of them). For the second position, we can choose any of the remaining 5 positions; hence, the number of different possible choices for

the pair is 6×5=30. Since both positions are filled with the same digit 1, we cannot distinguish them by their order, (e.g., the entry defined by the choice 1st, 2nd will be the same as the one defined by the choice 2nd, 1st), therefore, the actual number of different possible entries is twice as less as 30.
Answer: D

21. It takes Angie 2 hours round trip to walk a tour that contains a horizontal section and a slope section. On the way there she walks up hill on the slope section, and on the way back, she walks down hill on the same section. If Angie's speed is 4 km/h on the flat section, 3 km/h climbing and 6 km/h going down, what is the total distance of the tour (round trip)?
22.

A) Impossible to determine	B) 6 km	C) 7.5 km	D) 8 km	E) 10 km

Solution: Analysing the speeds of Angie on each of the three types of slopes, it is clear that:

- On a horizontal road, it takes 15 minutes to walk 1 km (since it takes 1hour (or 60 minutes) to walk 4 km, and 60/4=15).
- When climbing, it takes 20 minutes to walk 1 km (60/3=20).
- When going down, it takes 10 minutes to walk 1 km (60/6=10).

It is also clear that on the horizontal road, it will take 15+15=30 minutes to walk 1 km round trip, while on the slope, it will take a total of 20+10=30 minutes to walk 1 km round trip (one of the directions is up hill, the other one is down hill).
It follows that, regardless the slope, every kilometre of the distance between the starting and the ending point of the trip adds up 30 minutes to the total time for the round trip, namely, 2 hours or 120 minutes. Therefore, the distance is 120/30=4km. Angie walked this distance twice (round trip). So, the length of the trip is 8 km.
Answer: D.

23. The first digit of a 4 –digit number is equal to the number of zeroes in this number. The second digit of the number is equal to the number of digits 1, the third digit is equal to the number of digits 2, and the fourth digit represents the number of digits 3 in this number. How many numbers have this property?

A) 0	B) 2	C) 3	D) 4	E) 5

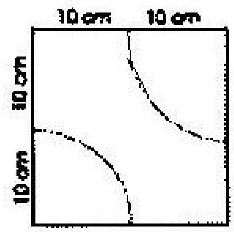
Solution: Denote by D1, D2, D3 and D4 the digits of the number. Since D1 is the number of zeros and we have a four-digit number, D1 has to be either 1, or 2, or 3.
Case 1: D1=3, so the number must contain three zeroes, but the number 3000 does not satisfy the requirements (the number of the digits 3 in D4 is not correct). So, there is no such number that starts with 3.
Case 2: D1=2. Then, we need two zeroes, but D3 cannot be 0, since we already used one digit 2, so the zeroes must be D2 and D4. The choice for D3 must be made from 1 or 2. It is easy to check that 2010 does not have the required property while 2020 has it.
Case 3: D1=1. Then, the number must contain one digit 0, which cannot be D2 (since we already used a digit 1). On the other hand, the digit D2 cannot be 1, because in this case there will be at least two digits 1 in the number and 2≠D2. The digit D2 also cannot be 3, because in this case we need to place three digits 1 in the number, which is not possible (only one undetermined digit out of the four is left). So, the number must be 12_ _ , where one of the D3 and D4 is 0 and the other one is 1. The number 1210 is the only one that satisfies all requirements in this case.
In conclusion, there are two numbers that have the required property.
Answer: B

24. The table 3×3 contains nine natural numbers (see the picture). Nick and Peter erased four numbers each so that the sum of the numbers erased by Nick was three times as great as the sum of the numbers erased by Peter. What number remained in the table?

4	12	8
13	24	14
7	5	23

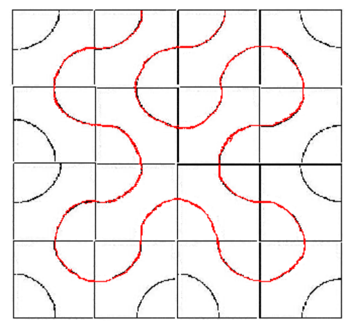
A) 4 B) 7 C) 14 D) 23 E) 24

Solution: The sum of the numbers in the table is 110. The sum of the erased numbers (S) is divisible by 4, because it is equal to 4 times the sum of the numbers erased by Peter. With respect to division by 4, the number left in the table and 110 must have the same remainder, since the remainder does not change when subtracting a multiple of 4 from 110. The remainder of 110 when divided by 4 is 2. The only number in the table that has a remainder of 2 when divided by 4 is 14.
Answer: C



25. On the picture, you can see a square tile, 20 cm × 20 cm. The design on the tile consists of two arcs of circles, as shown. If a table top with dimensions 80 cm × 80 cm is to be covered by these tiles so that some arcs connect in a curved line, what could be the maximum length of this line?

A) 75π B) 100π C) 105π D) 110π E) Impossible to determine



Solution: The longest curved line is composed of 22 arcs. One possible arrangement is shown on the figure. Each arc is a quarter of the circumference of a circle with a radius of 10 cm. Therefore the maximum length of the curved line is

$$22 \times \frac{1}{4} \times 2\pi \times 10 = 110\pi \text{ cm.}$$

Answer: D.

26. A three – digit number has been divided by 9. The sum of the digits of the result was 9 less than the sum of the digits of the number. For how many three-digit numbers would this be true?

A) 11 B) 5 C) 4 D) 2 E) 1

Solution: Denote the three-digit number by ABC and let S(ABC) denote the sum of the digits of the number ABC. From the given information, it is evident that ABC is a multiple of 9. Let Q be the result of the division ABC/9 (Q is a number between 12 and 111, including). From the divisibility hints, it is known that each number and the sum of its digits have the same remainder when divided by 9. ABC is a multiple of 9, therefore, S(ABC) is also a multiple of 9, hence, S(ABC)=9 or 18 or 27. Consequently, since S(Q)=S(ABC)-9, the number Q must be a multiple of 9, too, which means that the number ABC must be a multiple of 81.
On the other hand, since S(Q)≠0, S(ABC) ≠9.
If S(ABC)=27, ABC=999, Q=111 and S(ABC)-9≠S(Q). Therefore, S(ABC)=18 and S(Q)=9.
So far, we know that the number ABC is a multiple of 81 and the sum of its digits is 18.
Checking the three-digit multiples of 81, there are several possibilities for the number ABC: 486, 567, 648, 729, 891, or 972. The respective numbers Q are: 54, 63, 72, 81, 99, 108. Taking into consideration that S(Q)=9, one of the numbers, 891, must be excluded. As a result, there are five numbers satisfying the requirements.
Answer: B

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Bonus 1: *Al and Bill together weigh less than Charlie and Dan; Charlie ad Ed together weigh less than Frank and Bill. Which of the following sentences is certainly true?*

A) Al and Ed together weigh less than Frank and Dan;
B) Dan and Ed together weigh more than Charlie and Frank;
C) Dan and Frank together weigh more than Al and Charlie;
D) Al and Bill together weigh less that Charlie and Frank;
E) Al, Bill, and Charlie together weigh as much as Dan, Ed, and Frank.

Solution: Denote the weights of the six men by their initials, A,B, C, D, E, and F.
From the given information, the following inequalities hold:
$A+B < C+D$
$C+E < F+B$.
Using the inequalities for comparison, we can evaluate the total weight of Al, Bill, Charlie and Ed as follows: $(A+B)+(C+E) < (C+D)+(F+B)$. This inequality is true, since we replaced each of the addends in the sum on the left by an addend greater than it, thus, we increased the total sum. But in the last inequality, the weights of Bill and Charlie appear on both sides, so if we remove their weights from both sides, the new inequality will also hold: $A+E < D+F$. This is exactly the statement in A.
Answer: A

Bonus 2: *A positive integer number* ***n*** *has exactly 2 divisors, while the number* ***n+1*** *has exactly 3 divisors. How many divisors does the number* ***n+2*** *have?*

A) 2 *B) 3* *C) 4* *D) 5* *E) depends on the choice of* ***n***

Solution: If n has exactly 2 divisors, then it is a prime. Since $n+1$ has exactly 3 divisors, then $n+1$ is not a prime, but has to have only one prime factor, being its perfect square (like $9=3\times3$, for example). All numbers that have exactly three factors are odd, except the number 4, which is the only even such number. (A) If $n+1\neq4$, then $(n+1)$ is odd, therefore, n is even. But n must be prime and the only even prime number is 2, for which $(n+1)$ does not have three factors, so, it is not a possibility for $(n+1)$.
(B) If $n+1=4$, then $n=3$, $n+2=5$ and has 2 factors.
In conclusion, the only possibility is $n=3$, which implies that $(n+2)$ has exactly 2 factors.
Answer: A
