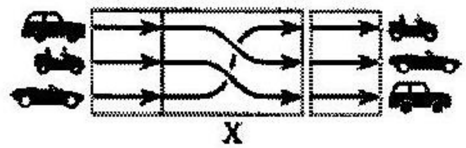
International Contest-Game
MATH KANGAROO
Canada, 2007

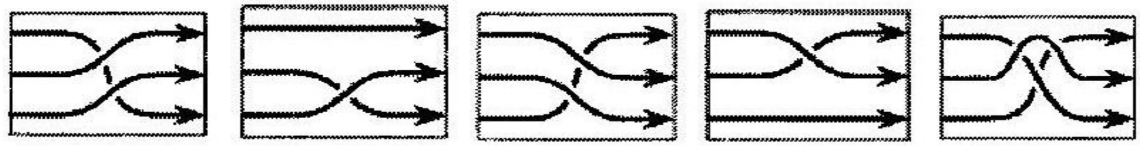
Grade 11 and 12

Part A: Each correct answer is worth 3 points.

1. Mike is building a race track. He wants the cars to start the race in the order presented on the left, and to end the race in the order presented to the right. Which of the elements given below should Mike use to replace the element X ?



A) B) C) D) E)



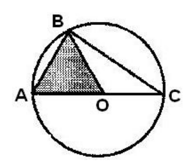
Solution: We need a piece where the first entering line exits third, the second entering line exits first and the third entering line exits second.
Answer: A

2. What is the value of $\dfrac{\sin 1°}{\cos 89°}$?

A) 0 B) tan 1° C) $1/\tan 1°$ D) 1/89 E) 1

Solution: Using the properties of the sine and the cosine, it follows that the numerator and the denominator are equal, so the value of the fraction is 1.
Answer: E

3. On the figure, the triangle ABC is inscribed in a circle with a centre O. The shaded area is equal to $\sqrt{3}$. What is the area of the triangle ABC?



A) $2\sqrt{3}$ B) 2 C) 5 D) 4 E) $4\sqrt{3}$

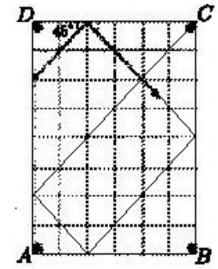
Solution: The shaded area is a half of the area of the triangle.
Answer: A

4. At the entrance examination to a university, a student must answer at least 80% of the questions correctly. So far, Peter has worked on 15 questions. He did not know the answer to 5 of them, but he was sure that he has answered the other 10 questions correctly. If he does not guess on the five questions he could not answer and answers all the remaining questions in test correctly, he will pass the test at exactly 80%. How many questions are there in the test?

A) 20 B) 25 C) 30 D) 35 E) 40

Solution: Let n be the number of remaining questions. Then, Peter would have answered (10+n) questions correctly, out of (15+n) questions in total. As this provides a result of 80%, the equation to solve is $\dfrac{10+n}{15+n} = 0.8$.
Answer: C.

5. A billiard ball is hit from a point on the vertical board of the table (close to the pocket D), as shown in the diagram. It meets the horizontal board at an angle of 45°, and continues following the direction of the arrow. Into which pocket will the ball fall?



A) A B) B C) C D) D E) neither of the pockets

Solution: Using the physics laws of reflection, it is clear that the ball will approach and will bounce from all walls at an angle of 45°. Therefore, all segments that trace the path of the ball will pass along diagonals of the grid squares. The completed path of the ball will be the one traced on the diagram.

Answer: C.

6. Some historians claim that the ancient Egyptians used a string with 2 knots to construct a right angle. If the length of the string is 12 m and one of the knots is at the point X 3 m from one of the ends, at what distance from the other end of the string should the second knot be put in order to obtain a right angle at X?

A) 3 B) 4 C) 5 D) 6 E) None of these

Solution: A right angle at X can be constructed by folding the string into a right-angled triangle, with the three sections between the knots being the sides of the triangle. Therefore, the question is equivalent to finding two numbers, say, *m* and *n* $(m<n)$, such that 3, *m*, and *n* satisfy the Pythagorean Theorem, that is $3^2+m^2=n^2$, and $3+n+m=12$. By noticing that the popular triple 3, 4, 5 has a sum of 12 one can quickly obtain the correct answer. Alternatively, the system of the above equations may be solved.

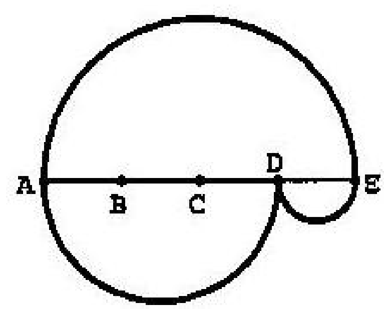
Answer: C.

7. An island is inhabited by knights and liars. Each knight always tells the truth and each liar always lies. Once an islander A, when asked about himself and another islander B, claimed that at least one of A and B is a liar. Which of the following sentences is true?

A) A is not able to make the above statement.
B) Both are liars.
C) Both are knights.
D) A is a liar while B is a knight.
E) B is a liar while A is a knight.

Solution: There are four possible combinations for A and B, the ones in the answers B, C, D, and E. If A is a liar, his statement would not be true, so, it means that none of the two is a liar, which contradicts to the assumption. If A is a knight, then he is telling the truth, and it is possible only if B is the liar.

Answer: E.



8. In the figure, AE is divided into four equal parts. Three semicircles are constructed taking AE, AD, and DE as diameters, and creating two paths from A to E, as shown. What is the ratio of the length of the curve above AE and the length of the curve below AE?

A) 1:2 B) 2:3 C) 2:1 D) 3:2 E) 1:1

Solution: Denote the length of AB by x. The length of the curve above AE is $\frac{1}{2}(2(2x)\pi) = 2x\pi$. The length of the curve below AE is $\frac{1}{2}(2(1.5x)\pi) + \frac{1}{2}(2(0.5x)\pi) = 2x\pi$.

Answer: E.

Part B: Each correct answer is worth 4 points.

9. Given a square ABCD with side 1, all squares are drawn that share at least two vertices with ABCD. What is the area of the region of all points covered by at least one of these squares?

A) 5 B) 6 C) 7 D) 8 E) 9

Solution: The squares that share at least two vertices with ABCD are two groups: (1) four squares congruent to it, constructed on each of the sides of ABCD; (2) four squares with a side equal to the diagonal of ABCD, whose centres are A,B,C and D, respectively. All squares that share three or more vertices with ABCD in fact coincide with ABCD. The required region has the form of an octagon and is composed of five squares with an area of 1 and four triangles, each of them being half of the squares. The total area, therefore, is 5+2=7.

Answer: C.

10. Angle β is 25 % less than angle χ and 50 % greater than angle α. Which of the following is true about angle χ?

A) It is 25% greater than α B) It is 50% greater than α C) It is 75% greater than α D) It is 100% greater than α E) It is 125% greater than α

Solution: The given information leads to the following relationships between the angles: $\beta=0.75\chi$; $\beta=1.5\alpha$, which are equivalent to: $\chi=4/3\beta$; $\alpha=2/3\beta$. Therefore, χ is twice as great as α, i.e. it is 100% greater than α.

Answer: D.

11. Given $2^{x+1} + 2^x = 3^{y+2} - 3^y$, where x and y are integers. What is the value of x?

A) 0 B) 3 C) -1 D) 1 E) 2

Solution: The given equation can be transformed as follows:
$2^{x+1} + 2^x = 3^{y+2} - 3^y \Leftrightarrow 2^x(2+1) = 3^y(3^2-1) \Leftrightarrow 2^{x-3} = 3^{y-1}$. The last equation is possible only if x-3=y-1=0. Hence, x=3, y=1.

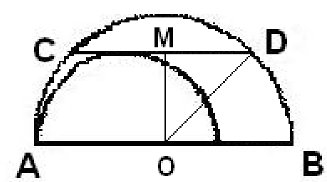
Answer: B.

12. What is the value of *cos 1°+cos 2°+cos 3°+…+cos 358°+cos 359°*?

A) 1 B) π C) 0 D) 10 E) -1

Solution: The sum contains 359 addends. Using the properties of the cosine function, *cos 1°+ cos 181°= cos 2°+ cos 182°=…= cos 179°+ cos 359°=0,* and *cos 180°=-1*. Therefore, the value of the given sum is (-1).

Answer: A



13. Two semicircles are constructed, as shown in the figure. The chord CD is parallel to the diameter AB of the greater semicircle and touches the smaller semicircle. If the length of CD is 4, what is the area of the shaded region?

A) π B) 1.5π C) 2π D) 3π E) more information needed

Solution: Let O be the centre of the great semicircle and OM is the perpendicular from O to CD. It is surprising that the area of the region between the two semicircles seems to be dependent on the length of the chord CD only, and independent on the radius of any of the semicircles. The key to justifying this statement is the triangle MOD, where MO is the radius, r, of the small semicircle, OD is the radius, R, of the great semicircle, and MD is 2, as it is the half of CD. By the Pythagorean triangle applied for triangle MOD, $R^2 - r^2 = 2^2$. On the other hand, the area of the shaded region is $\frac{1}{2}\left(\pi R^2 - \pi r^2\right) = \frac{1}{2}\pi\left(R^2 - r^2\right) = \frac{1}{2}\pi\left(2^2\right) = 2\pi.$

Answer: C.

14. The sum of five consecutive integers is equal to the sum of the next three consecutive integers. What is the greatest of these eight numbers?

A) 4 B) 8 C) 9 D) 11 E) something else

Solution: Denote the greatest of the integers by x. Then, the following equation holds: *x+(x-1)+(x-2)=(x-3)+(x-4)+(x-5)+(x-6)+(x-7).* Its solution is x=11.

Answer: D

15. Thomas was born on his mother's 20th birthday, and so they share birthdays. How many times will Thomas's age be a divisor of his mother's age if they both live long lives?

A) 4 B) 5 C) 6 D) 7 E) 8

Solution: If Thomas's age is x, then his mother's age is (x+20). It is required that (x+20) is a multiple of x. This is possible only if 20 is a multiple of x, which implies that x=1, 2, 4, 5, 10, or 20. Therefore, Thomas's age will be a divisor of his mother's age 6 times.

Answer: C.

16. Consider a sphere of radius 3 with centre at the origin of a Cartesian co-ordinate system. How many points on the surface of this sphere have integer co-ordinates?

A) 30 B) 24 C) 12 D) 6 E) 3

Solution: We have to count the points (x, y, z), for which the coordinates x, y, z, are integer numbers and satisfy the equation $x^2 + y^2 + z^2 = 3^2$. There are only two sums of perfect squares that equal to 9 (9=9+0+0=1+4+4). So, the only groups of integer numbers that might be coordinates of the points are ±1, ±2, ±2 or ±3, 0, 0. There are 24 points whose coordinates are the numbers from the first group. In fact, there are 3 possibilities to choose the position of the 1 (or -1). For each of them, there are 8 possible arrangements of the signs + and -, so there are 3x8=24 arrangements for the coordinates. There are 6 points in the second group, since there are three possibilities to choose the position of the non-zero digit, and for each of them, there are two choices of the sign.
In conclusion, the required number of points is 30.

Answer: A.

Part C: Each correct answer is worth 5 points.

17. Which of the following numbers cannot be written as $x+\sqrt{x}$, if x is an integer number?

A) 870 B) 110 C) 90 D) 60 E) 30

Solution: Since the proposed answers are integer numbers, it follows that $\sqrt{x}$ is an integer number, hence, x is a perfect square. On the other hand, $x+\sqrt{x}=\sqrt{x}\,(\sqrt{x}+1)$, which is a product of two consecutive positive integer numbers. The number 60 is the only one of the above numbers that cannot be represented in this way.

Answer: D.

18. If $f(x) = \frac{2x}{3x+4}$ and $f(g(x)) = x$, what is the equation of the function *g(x)*?

A) $g(x)=\dfrac{3x+4}{2x}$ B) $g(x)=\dfrac{3x}{2x+4}$ C) $g(x)=\dfrac{2x+4}{4x}$ D) $g(x)=\dfrac{4x}{2-3x}$ E) Other answer

Solution: Students familiar with the concept of inverse function will notice that g(x) is the inverse function of f(x). To find its equation, denote f(x) by y, solve the equation for x in terms of y and replace x by g(x) and y by x:

$y=\dfrac{2x}{3x+4}\Leftrightarrow 3xy+4y=2x\Leftrightarrow x=\dfrac{4y}{2-3y}$, so $g(x)=\dfrac{4x}{2-3x}$.

Alternatively, one can check which of the equations for g(x) in the answers satisfies the equation $f(g(x))=x$.

Answer: D.

19. What is the measure of the acute angles of a rhombus, if its side is the geometrical mean of the diagonals? *(Note: The number C=* $\sqrt{A\times B}$ *is called Geometrical mean of the numbers* A and B).

A) 15° B) 30° C) 45° D) 60° E) 75°

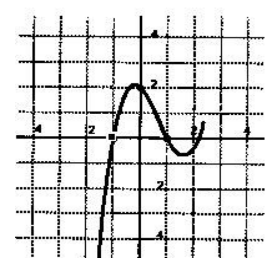
Solution: Denote the side of the rhombus ABCD by *a*, and the diagonals as follows:
AC= *m*, BD=*n*. From the given, it follows that $a=\sqrt{m\times n}\Leftrightarrow a^2=mn.$ Applying the Cosine Law for triangles ABC and ABD and the fact that angles A and B are supplementary (thus, their cosines are opposite numbers), it follows that

$m^2=a^2+a^2-2a^2\cos A$

$n^2=a^2+a^2-2a^2\cos B=a^2+a^2+2a^2\cos A$

$a^4=m^2n^2=(2a^2(1-\cos A))(2a^2(1+\cos A))\Leftrightarrow 1-\cos^2 A=\dfrac{1}{4}\Leftrightarrow \sin^2 A=\dfrac{1}{4}\Leftrightarrow A=30°.$

Answer: B.



20. The graphic on the right is a piece of the graph of the function $f(x)=ax^3+bx^2+cx+d$. It passes through the points (-1, 0), (0, 2) and (1, 0). What is the value of *b*?

A) -4 B) -2 C) 0 D) 2 E) 4

Solution: Since the point (0, 2) is on the graph, it follows that f(0)=d=2.
Next, *f(-1)= -a+b-c+2=0*, and *f(1)=a+b+c+2=0.* Adding the two last equations, we get *2b+4=0,* therefore, *b=-2.*

Answer: B

21. For how many real numbers *a* does the quadratic equation $x^2+ax+2007=0$ have two integer roots?

A) 3 B) 4 C) 6 D) 8 E) another answer

Solution: Let m and n be the roots of the quadratic equation. Hence, they satisfy the following two equations: *m+n=-a, mn=2007* (according to the properties of the roots of a polynomial equation, also known as Viette's formulas). We are interested in integer roots only; therefore, we must consider all possible representations of 2007 as a product of two integer numbers. In fact, there are exactly 6 such representations:
2007=1×2007=(-1) × (-2007)=3×669=(-3) ×(-669)=9×223=(-9) ×(-223). The respective values of *a* for each of them are: *a=2008, -2008, 672, -672, 232, -232.*

Answer: C.

22. On a party, five friends are going to give each other gifts in such a way that everybody gives one gift and receives one gift (of course, no one should receive their own gift). In how many ways is this possible?
23.

A) 5 B) 10 C) 44 D) 50 E) 120

Solution: Denote the friends by A, B, C, D and E. The set of possible distributions of gifts can be split into two non-overlapping subsets: (1) Among the 5 friends, there are two who exchanged their gifts (i.e. A gets the present of B and B gets the present of A). (2) Among the 5 friends, there is no such pair that exchanged their gifts. We will count the possibilities for each of the two subsets.

(1) Assume that two of the friends, say A and B, exchanged their gifts. It follows that among the other three friends, there weren't a pair that also exchanged their gifts, because otherwise, the fifth person will have nobody to give his/her gift to, which is not allowed. For the three friends, C, D, and E, there are only two possibilities to exchange gifts: C gives to D, D gives to E, E gives to C or C gives to E, E gives to D, D gives to C. It follows that the number of the possibilities in this subset can be calculated by doubling the possibilities to choose two friends out of the five friends (which is equal to ${}_5^2C = \binom{5}{2} = \frac{5\times 4}{1\times 2} = 10$). Therefore, there are 20 possible ways to exchange gifts in a way that two friends would exchange gifts between themselves.

(2) Assume that no two friends exchanged their own gifts. Without loss of generality, let A gives a gift to B. Then, B cannot give the gift to A, so, B gives the gift to somebody of C, D, or E. Assume, B gives the gift to C. Similarly, C must give to either D or E, assume, to D, and, finally, D gives to to E and E gives to A, hence, the exchange is only possible " around the circle", where the five friends are arranged in any order. Therefore, the number of possibilities in this subset is equal to the number of different arrangements of five elements around a circle. Note that two arrangements around a circle are different if and only if there is at least one element that has at least one neighbour that is different. To count the number of there arrangements, first, consider all possible arrangements of five elements in a row (there are 5!=120 such arrangements). Then, note that, in terms of the definition of different arrangements around the circle, there will be groups of arrangements that are not distinct, for instance, ABCDE is equivalent to BCDEA, CDEAB, DEABC, and EABCD. More specifically, for each permutation of the five elements, there will be four more that are equivalent to it. Therefore, the number of different arrangements of five elements around a circle is 120/5=24.

In conclusion, the number of possible ways the five friends can exchange gifts is 20+24=44.

Answer: C.

24. What is the sum $\frac{1}{2\sqrt{1}+1\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \frac{1}{4\sqrt{3}+3\sqrt{4}} + ... + \frac{1}{100\sqrt{99}+99\sqrt{100}}$?

A)999/1000 B) 99/100 C) 9/10
D) 9 E)1

Solution: Each of the fractions in the sum is of the form $\frac{1}{n\sqrt{n-1}+(n-1)\sqrt{n}}$, where *n=2, 3, …100.* We will transform this fraction as follows:

$$\frac{1}{n\sqrt{n-1}+(n-1)\sqrt{n}}=\frac{1}{\left(n\sqrt{n-1}+(n-1)\sqrt{n}\right)}\cdot\frac{n\sqrt{n-1}-(n-1)\sqrt{n}}{\left(n\sqrt{n-1}-(n-1)\sqrt{n}\right)}=$$

$$=\frac{n\sqrt{n-1}-(n-1)\sqrt{n}}{\left(n^2(n-1)-(n-1)^2n\right)}=\frac{\sqrt{n}\sqrt{n-1}\left(\sqrt{n}-\sqrt{n-1}\right)}{n(n-1)(n-n+1)}=$$

$$=\frac{\sqrt{n}\sqrt{n-1}\left(\sqrt{n}-\sqrt{n-1}\right)}{n(n-1)(n-n+1)}=\frac{\left(\sqrt{n}-\sqrt{n-1}\right)}{\sqrt{n}\sqrt{n-1}}=\frac{1}{\sqrt{n-1}}-\frac{1}{\sqrt{n}}.$$

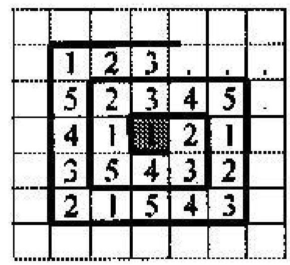
Next, applying the result for *n=2, 3, …100*, the given sum becomes:

$$S=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}+....+\frac{1}{\sqrt{98}}-\frac{1}{\sqrt{99}}+\frac{1}{\sqrt{99}}-\frac{1}{\sqrt{100}}\Leftrightarrow$$

$$S=\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{100}}=\frac{9}{10}.$$

Answer: C.

24. The digits of the sequence 123451234512345… fill the cells in the table in a spiral-like manner beginning from the marked cell (see the figure). Which digit is written on the cell placed 100 cells above the marked one?



A) 1 B) 2 C) 3 D) 4 E) 5

Solution: Suppose the table has been filled in the specified manner and the last cell filled is the n[th] cell above the marked one. Thus, the part of the table that is filled has a shape of a square, whose side is (2n+1) units long, without the last *n* cells from the upper row, just to the right of the marked cell. Then, the total number of the filled cells in the shape is

$(2n+1)^2-n$, therefore, the required number is the ($(2n+1)^2-n$)-th term of the sequence 123451234512345…. (it is, in fact, the last number written).

For n=100, $(2n+1)^2-n=40301$. To find the last written digit, one must calculate the marked cell is 1.

Answer: A.

…………………

Bonus 1: *The increasing sequence 1, 3, 4, 9, 10, 12, 13, … includes all the powers of 3 and all the numbers that can be written as the sum of different powers of 3. What is the 100th element of the sequence?*

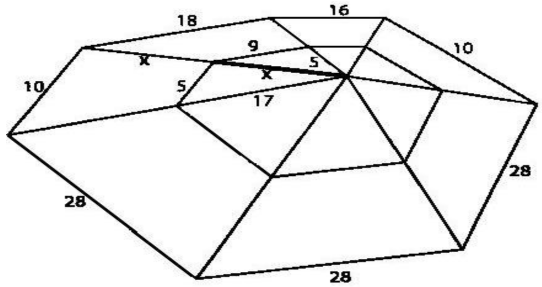
A) 150 B) 981 C) 1234 D) 2401 E) 3^{100}

Solution: First, let us recall that $3^k>3^{k-1}+3^{k-2}+....+3^2+3^1+3^0$. This can be easily justified using the formula for the geometric series for the right-hand side or from the identity

$x^k-1=(x-1)(x^{k-1}+x^{k-2}+....+x^2+x^1+x^0)$ applied for x=3. Hence, each of the terms in the above sequence has a unique representation as a sum of several powers of 3.

Next, for each number, let us define a binary sequence of 0s and 1s that represents whether a specific power of 3 is included in the number's representation as a sum or not. From right to left, the digits of the binary sequence correspond to the powers 3^0, 3^1, 3^2,…., etc. For instance, $10=9+1=3^2+3^0$, so the binary sequence defined for 10 is 101. It is clear now that the binary sequence representing the 100th number of the original sequence is the number 100 written in a Base 2 numerical system, i.e., 1100100. It means that the addends that compose the 100th element are 3^2, 3^5, and 3^6, hence, the 100th element is 9+243+729=981.

Answer: B.



Bonus 2: *A mathematically skilled spider spins a web and some of the strings have lengths as shown in the picture. If x is an integer, determine the value of x.*

A) 11 *B)*13 *C)* 15 *D)* 17 *E) 19*

Solution: The key to solving this problem is the triangle inequality that states that three segments, a, b, and c, can form a triangle if and only if each of them is greater than the difference of the other two and smaller than the sum of the other two segments.

Assuming (without loss of generality) $a \geq b \geq c$, the inequalities that represent this statement are: *b+c>a>b-c; a+c>b>a-c; a+b>c>a-b.* In the net, there are two triangles with a side x. Applying the triangle inequality for each of them, it follows that x<5+9=14 and x>17-5=12. The only integer number that satisfies both conditions is 13, therefore, x=13.

Answer: B.

Bonus 3: *Ann, Belinda and Charles are throwing a die. Ann wins if she throws a 1, 2, or 3; Belinda wins if she throws a 4 or 5; Charles wins if he throws a 6. The die rotates from Ann to Belinda to Charles to Ann, etc., until one player wins. Calculate the probability that Charles wins.*

A) 1/6 *B)*1/8 *C)* 1/11 *D)* 1/13 *E) It is impossible for Charles to win*

Solution: Denote by A, B, C the events of winning the game by Ann, Belinda or Charles, respectively, and by $\overline{A}, \overline{B}, \overline{C}$ - the events of not winning the game by each of them.

The probability of winning the game by Charles on his first turn is equal to

$P(\overline{A}) \times P(\overline{B}) \times P(C) = \frac{1}{2} . \frac{4}{6} . \frac{1}{6} = \frac{1}{18}$. If Charles wins the game on his second turn, it means that he did not win on his first turn and Ann and Belinda have not won the game after their second turn. Then the probability of Charles winning the game at the second round of the game is $\left(\frac{1}{2}\right)^2 . \left(\frac{4}{6}\right)^2 . \frac{5}{6} . \frac{1}{6} = \frac{5}{18} . \frac{1}{18}$. If Charles wins the game after n turns, it means that all of them have not won after (n-1) rounds, and on the last round, Ann and Belinda did not win while Charles won. For each of the first (n-1) rounds, the probability is $\left(\frac{1}{2}\right) . \left(\frac{4}{6}\right) . \frac{5}{6} = \frac{5}{18}$, and for the last round, the probability is $\left(\frac{1}{2}\right) . \left(\frac{4}{6}\right) . \frac{1}{6} = \frac{1}{18}$. So, the probability that Charles wins the game at the n-th round is $\left(\frac{5}{18}\right)^{n-1} . \frac{1}{18}$. Since the number of rounds in the game is not limited, the overall probability of Charles winning the game is an infinite sum:

$P = \frac{1}{18} + \left(\frac{5}{18}\right)^1 . \frac{1}{18} + \left(\frac{5}{18}\right)^2 . \frac{1}{18} + \left(\frac{5}{18}\right)^3 . \frac{1}{18} + ... + \left(\frac{5}{18}\right)^{n-1} . \frac{1}{18} + ...$ Using the formula for the infinite geometric series with a ratio less than 1, we calculate: $P = \frac{1}{18}\left(\frac{1}{1 - \frac{5}{18}}\right) = \frac{1}{13}$.